Single Particle Motion

C ontents

Plasmas somewhere between fluids and single particles. Here consider single particles.

Uniform E and B

E = 0 - guiding centers

$$m\frac{d\bar{\mathrm{v}}}{dt} = q\bar{\mathrm{v}} \times \bar{B}$$

Take $\bar{B} = B\hat{z}$. The

$$m\dot{\mathrm{v}}_x = qB\mathrm{v}_y; \quad m\dot{\mathrm{v}}_y = -qB\mathrm{v}_x; \quad m\dot{\mathrm{v}}_z = 0$$

$$\ddot{\mathrm{v}}_x = \frac{qB}{m}\dot{\mathrm{v}}_y = -\left(\frac{qB}{m}\right)^2 v_x$$

$$\ddot{\mathrm{v}}_y = -\frac{qB}{m}\dot{\mathrm{v}}_x = -\left(\frac{qB}{m}\right)^2 v_y$$

Harmonic oscillator at cyclotron frequency

$$\omega_c = \frac{|q|B}{m}$$

i.e.

$$\mathrm{v}_{x,y} = \mathrm{v}\; e^{\pm i\omega_c t + i\delta_{x,y}}$$

$\pm$ denotes sign of q. Choose phase δ so that

$$\mathrm{v}_x = \mathrm{v}\; e^{i\omega_c t} = \dot{x}$$

where v is a positive constant, speed in plane perpendicular to vector B. Then

$$\mathrm{v}_y = \frac{m}{qB}\dot{\mathrm{v}}_\mathrm{x} = \pm\frac{1}{\omega_\mathrm{c}}\dot{\mathrm{v}}_\mathrm{x} = \pm\mathrm{v}\; e^{i\omega_c t} = \dot{y}$$

Integrate

$$x - x_0 = -\mathrm{i}\frac{\mathrm{v}}{\omega_\mathrm{c}} e^{i\omega_c t}; \quad y - y_0 = \pm\frac{\mathrm{v}}{\omega_\mathrm{c}} e^{i\omega_c t}$$

Define Larmor radius

$$r_L = \frac{\mathrm{v}}{\omega_c} = \frac{m\mathrm{v}}{|q|B}$$

Taking real parts (fn + complex conj)/2) gives

$$x - x_0 = r_L \sin(\omega_c t); \quad y - y_0 = \pm r_L \cos(\omega_c t)$$

i.e. a circular orbit around B. Ions and electrons circulate in opposite directions. The sense is such that the B field generated by the particle always tends to reduce the external B, i.e. plasmas are diamagnetic. Electrons have smaller Larmor radii than ions.

Definition of guiding center

Defined as

$$\bar{r}_{gc} = r - \bar{R}_c$$

r is position vector of particle and R_c is radius of curvature, which is a vector from the position of the particle to the center of gyration. In the plane perpendicular to B vector write in terms of momentum vector p:

$$m\omega_c^2 \bar{R}_c = q\bar{\mathrm{v}} \times \bar{\mathrm{B}}$$

Now $\omega_c^2 = q^2 B^2 / m^2$, so

$$\bar{R}_c = \frac{\bar{p} \times \bar{\mathrm{B}}}{qB^2}$$

and

$$\bar{r}_{gc} = \bar{r} - \frac{\bar{p} \times \bar{\mathrm{B}}}{qB^2}$$

Now consider a collision in which a force f is applied to the particle in a direction perpendicular to the vector B, and this force is f >> the Lorentz force. Let the field be homogeneous. Let the impact time be very short, << the Larmor frequency. At the collision the momentum p changes a lot but the particle position vector r does not. The momentum p changes from p to p' = p + Δp, where

$$\Delta\bar{p} = \int_t^{t+\Delta t} \bar{f}\,dt$$

Now the guiding center must move by an amount

$$\Delta\bar{r}_{gc} \approx \frac{\Delta\bar{p} \times \bar{\mathrm{B}}}{qB^2}$$

Generalizing to a continuous force

$$\frac{d}{dt}\bar{r}_{gc} = \frac{d}{dr}\bar{r} + \frac{\frac{d}{dt}\bar{p} \times \bar{\mathrm{B}}}{qB^2}$$

now use $\frac{d\bar{p}}{dt} = q\bar{\mathrm{v}} \times \bar{\mathrm{B}} + \bar{\mathrm{F}}$

where F is a non magnetic force. Then

$$\frac{d\bar{r}_{gc}}{dt} = \mathrm{v} + \frac{\left(\bar{F} + q\bar{\mathrm{v}} \times \bar{\mathrm{B}}\right) \times \bar{\mathrm{B}}}{qB^2}$$

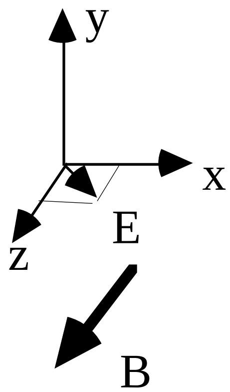
$$= \mathrm{v} + \frac{\left(\bar{F} \times \bar{B} + q\left(\bar{\mathrm{v}} \times \bar{\mathrm{B}}\right) \times \bar{\mathrm{B}}\right)}{qB^2}$$

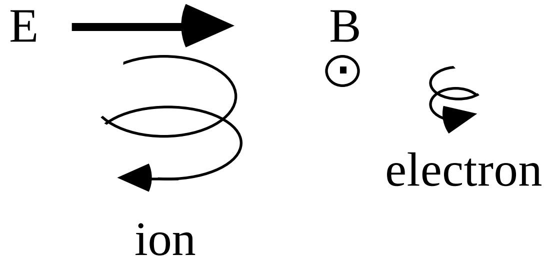
$$= \bar{\mathrm{v}}_{\parallel} + \frac{\bar{F} \times \bar{B}}{qB^2}$$

(expand triple vector product and use $\bar{\mathrm{v}} = \bar{\mathrm{v}} \;\; + \bar{\mathrm{v}}_{\parallel}$

i.e. if the force is continuous, the guiding center motion can be viewed as a continuous series of small impacts.

E ≠ 0





Will find motion is sum of usual circular Larmor orbit plus a drift of the 'guiding center'. Choose E in the x-z plane so that $E_y = 0$. As before, z component of velocity is unrelated to the transverse components and can be treated separately.

$$m\frac{d\overline{\mathrm{v}}}{dt} = q\left(\overline{E} + \overline{\mathrm{v}} \times \overline{B}\right)$$

z component

$$m\frac{d\mathrm{v}_z}{dt} = q\left(E_z\right); \quad \mathrm{v}_z = \frac{qE_z}{m}t + \mathrm{v}_{z0}$$

i.e. acceleration along the vector B. Transverse components give

$$\dot{\mathrm{v}}_x = \frac{q}{m}E_x \pm \omega_c \mathrm{v}_y; \quad \dot{\mathrm{v}}_y = 0 \mp \omega_c \mathrm{v}_x$$

Differentiate

$$\ddot{\mathrm{v}}_x = -\omega_c^2 \mathrm{v}_x$$
$$\ddot{\mathrm{v}}_y = \mp\omega_c\left(\frac{q}{m}E_x \pm \omega_c \mathrm{v}_y\right) = -\omega_c^2\left(\frac{E_x}{B} + \mathrm{v}_y\right)$$

i.e.

$$\frac{d}{dt^2}\mathrm{v}_x = -\omega_c^2 \mathrm{v}_x$$
$$\frac{d}{dt^2}\left(\mathrm{v}_y + \frac{E_x}{B}\right) = -\omega_c^2\left(\mathrm{v}_y + \frac{E_x}{B}\right)$$

i.e. just like E = 0 case except replace v_y by $\mathrm{v}_y + E_x/B$. Therefore solution is

$$v_x = v\ e^{i\omega_c t}$$
$$v_y = \pm i v\ e^{i\omega_c t} - \frac{E_x}{B}$$

i.e. there is a drift in the -y direction. Electrons and ions drift in the same direction.

More generally, can obtain an equation for the 'guiding center' drift v_{gc}. Omit dv/dt terms, as we know this just gives the Larmor orbit and frequency. Then

$$\left(\bar{E} + \bar{v} \times \bar{B}\right) = 0$$
$$\bar{E} \times \bar{B} = -\left(\bar{v} \times \bar{B}\right) \times \bar{B} = \bar{B} \times \left(\bar{v} \times \bar{B}\right)$$
$$= vB^2 - \bar{B}\left(\bar{v} \cdot \bar{B}\right)$$

(Use

$$\bar{a} \times \left(\bar{b} \times \bar{c}\right) = \bar{b}\left(\bar{a} \cdot \bar{c}\right) - \bar{c}\left(\bar{a} \cdot \dot{b}\right)$$
$$a = B; b = v; c = B: \qquad)$$
$$\bar{B} \times \left(\bar{v} \times \bar{B}\right) = \bar{v}\left(\bar{B} \cdot \bar{B}\right) - \bar{B}\left(\bar{B} \cdot \bar{v}\right)$$

transverse component gives the electric field drift of the guiding center"

$$\bar{v}_{gc} = \frac{\bar{E} \times \bar{B}}{B^2}$$, independent of q, m, ω_c.

This is because on first half of orbit a particle gains energy from E field, so velocity and Larmor radius increase. But on second half of orbit the particle loses energy and Larmor radius decreases. The difference in Larmor radius causes the drift.

gravitation

Generalize by replacing electrostatic by a general force . Then

$$\bar{v}_{gc} = \frac{1}{q} \frac{\bar{F} \times \bar{B}}{B^2}$$

e.g. for gravity

$$\bar{v}_{gc} = \frac{m}{q} \frac{\bar{g} \times \bar{B}}{B^2}$$

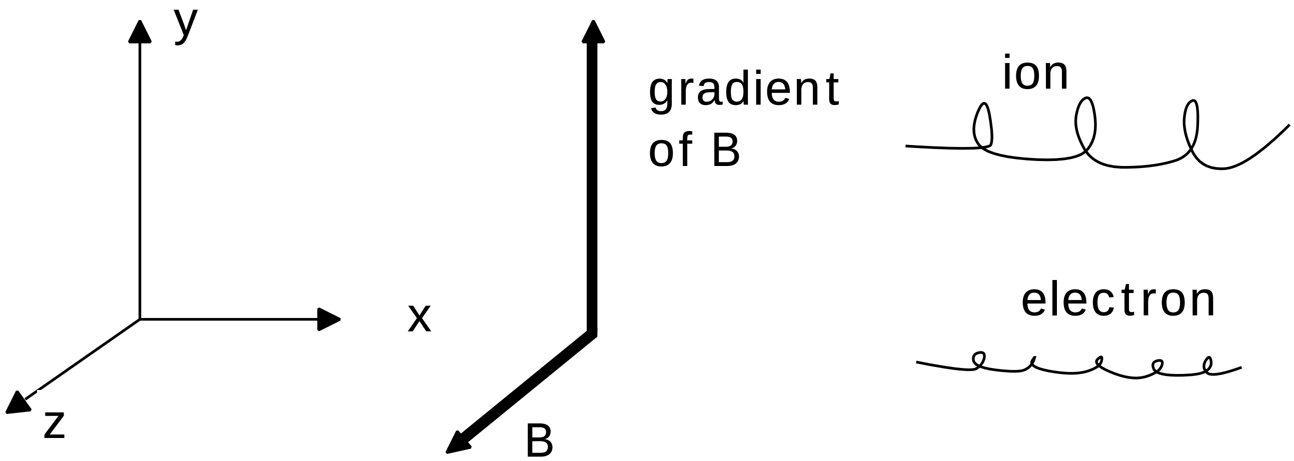
Note this is charge dependent, but very small. Therefore under gravity the charges drift and produce a net current

$$\mathrm{j} = \sum n_i q_i v_i = n(m+M)\frac{\bar{g}\times\bar{B}}{B^2}$$

Non Uniform B

Need to expand in a small parameter; orbit theory

'grad B' drift $\bar{B}$ $\bar{B}$



Take straight field lines in y direction, but with a gradient in their density, . Anticipate result: gradient in |B| causes the Larmor radius to be different, If gradient in y direction then smaller radius at top than bottom, , which will give a drift opposite for e and i, perpendicular to both the field and its gradient .

Average the Lorentz force over a gyration. Note $\langle F_x \rangle = 0$ as equal time moving up and down. To calculate average $\langle F_y \rangle$ use undisturbed orbit. Expand the field vector around the point x_0, y_0 so that

$$\bar{B} = \bar{B}_0 + (\bar{r}\cdot \quad)\bar{B} + ..$$

$$B_z = B_0 + y\frac{\partial B_z}{\partial y} + ..$$

$$F_y = -q\mathrm{v}_x B_z(y) = -q\mathrm{v}\quad \cos(\omega_c t)\left[B_0 + y\frac{\partial B}{\partial y} + \ldots\right]$$

$$-q\mathrm{v}\quad \cos(\omega_c t)\left[B_0 \pm r_L \cos(\omega_c t)\frac{\partial B}{\partial y} + \ldots\right]$$

This assumes $r_L/L << 1$, where L is scale of $\partial B/\partial y$. Then

$$\langle F_y \rangle = \mp\frac{1}{2}q\mathrm{v}\quad r_L\left(\frac{\partial B}{\partial y}\right)$$

and the guiding center drift is

$$\mathrm{v}\ _{B} = \pm\frac{1}{2}\mathrm{v}\ r_L\frac{\bar{B}\times\quad B}{B^2}$$

with ± standing for the sign of the charge.

Curvature drift

Assume lines of force have constant radius of curvature R_c. Take |B| constant. This does not satisfy Maxwell's equations, so we will always add the grad B drift to the answer we are about to get. Let $\mathrm{v}_{\parallel}^2$ be the average of the square of the (random) velocity along the vector B. then the average centrifugal force is

$$\bar{F}_{cf} = \frac{m\mathrm{v}_{\parallel}^2}{\mathrm{R_c}}\hat{r} = m\mathrm{v}_{\parallel}^2\frac{\bar{R}_c}{R_c^2}$$

Therefore

$$\bar{\mathrm{v}}_R = \frac{1}{q}\frac{\bar{F}_{cf}\times\bar{B}}{B^2} = \frac{m\mathrm{v}_{\parallel}^2}{qB^2}\frac{\bar{R}_c}{R_c^2}$$

Now compute the associated grad-B drift. Use cylindrical coordinates. $\times\bar{B} = 0$ in vacuum.. $\bar{B}$ has only a θ component, and B only a radial component , so

$$(\quad\times\bar{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 0; \quad B_\theta \quad \frac{1}{r}$$

and

$$|B| \quad \frac{1}{R_c}; \quad \frac{|B|}{|B|} = -\frac{\bar{R}_c}{R_c^2}$$

so that

$$v_{\nabla B} = \mp\frac{1}{2} v\, r_L \frac{\bar{B}\times|B|\bar{R}_c}{B^2R_c^2} = \pm\frac{1}{2}\frac{v^2}{\omega_c}\frac{\bar{R}_c\times\bar{B}}{BR_c^2} = \frac{1}{2}\frac{m}{q}v^2\frac{\bar{R}_c\times\bar{B}}{B^2R_c^2}$$

Adding the curvature term to this gives

$$v_R + v_{\nabla B} = \frac{m}{q}\frac{\bar{R}_c\times\bar{B}}{B^2R_c^2}\left(\frac{v^2}{2} + v_\parallel^2\right)$$

Now for a Maxwellian plasma $v_\parallel^2 = \frac{1}{2}v^2 = k_bT$ (because the perpendicular component has 2 degrees of freedom), so that

$$\bar{v}_{R+\nabla B} = \pm\frac{v_{th}^2}{R_c\omega_c}\hat{y} = \pm\frac{r_{L,atv_{th}}}{R_c}\hat{y}$$

$$where\ \hat{y} = \frac{\bar{R}_c\times\bar{B}}{R_c\times B}$$

i.e. dependent on q but not m.

Grad - B drift, $\nabla B \parallel \bar{B}$

Now consider B field along z, with magnitude dependent on z. Consider symmetric case; $\partial/\partial\theta = 0$. Then lines of force diverge and converge, and $B_r \neq 0$. From $\nabla\cdot\bar{B} = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

Suppose $\partial B_Z/\partial z$ is given at r = 0, and is not strongly dependent on r, then

$$rB_r = -\int_0^r r\frac{\partial B_z}{\partial r}dr \approx -\frac{1}{2}r^2\left[\frac{\partial B_z}{\partial r}\right]_{r=0}$$

$$B_r \approx -\frac{1}{2}r\left[\frac{\partial B_z}{\partial r}\right]_{r=0}$$

Variation of |B| with r causes a grad-B drift of the guiding centers about the axis of symmetry. However there is no radial grad - B drift because $\partial B/\partial\theta = 0$.

The component of the Lorentz force are

$$F_r = q\left(v_\theta B_z - v_z B_\theta\right) = q\left(v_\theta B_z\right)$$
$$F_\theta = q\left(-v_r B_z + v_z B_r\right)$$
$$F_z = q\left(v_r B_\theta - v_\theta B_r\right) = q\left(-v_\theta B_r\right)$$

The terms $F_r = q\left(v_\theta B_z\right)$; F_θ *part a* $= q\left(-v_r B_z\right)$ give the Larmor gyration. The term F_θ *part b* $= q\left(v_z B_r\right)$ vanishes on axis, and off axis it causes a drift in the radial direction. This makes the guiding centers follow the 'lines of force'. The final term, $F_z = q\left(-v_\theta B_r\right)$ is written as

$$F_z = \frac{1}{2} q v_\theta r \left(\frac{\partial B_z}{\partial z}\right)$$

Average over a gyration period. Consider a particle with guiding center on axis. The $v_\theta = v$; $r = r_L$

$$\langle F_z \rangle = \mp \frac{1}{2} q v\ r_L \left(\frac{\partial B_z}{\partial z}\right) = \mp \frac{1}{2} q \frac{v^2}{\omega_c} \left(\frac{\partial B_z}{\partial z}\right) = -\frac{1}{2} \frac{mv^2}{B} \left(\frac{\partial B_z}{\partial z}\right)$$

Define magnetic moment

$$\mu = \frac{1}{2} \frac{mv^2}{B}$$

Then in general the parallel force on a particle is given in terms of the element ds along the vector B:

$$\bar{F}_\parallel = -\mu \frac{\partial B}{\partial s} = -\mu \nabla_\parallel B$$

Note for a current loop area A current I then

$$\mu = IA = \frac{e\omega_c}{2\pi} \pi r_L^2 = \frac{e\omega_c}{2\pi} \frac{\pi v^2}{\omega_c^2} = \frac{1}{2} \frac{ev^2}{\omega_c} = \frac{mv^2}{2B}$$

Invariance of μ.

Consider parallel equation of motion

$$m \frac{dv_\parallel}{dt} = -\mu \frac{\partial B}{\partial s}$$
$$m v_\parallel \frac{dv_\parallel}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v_\parallel^2\right) = -\mu \frac{\partial B}{\partial s} \frac{ds}{dt} = -\mu \frac{dB}{dt}$$

where dB/dt is the variation of B seen by the particle. (B itself is or can be constant). Now conserve energy

$$\frac{d}{dt}\left(\frac{1}{2} m v_\parallel^2 + \frac{1}{2} m v_\perp^2\right) = \frac{d}{dt}\left(\frac{1}{2} m v_\parallel^2 + \mu B\right) = 0$$

Then

$$\frac{d}{dt}(\mu B) - \mu \frac{dB}{dt} = 0; \quad \frac{d\mu}{dt} = 0$$

Magnetic mirrors, loss cone

Finally we have the magnetic mirror. A particle moves from weak to strong field. B increases, so $v_\perp$ must increase to conserve μ. Then $v_\parallel$ must decrease to conserve energy. If B is high enough at some place along the trajectory, then $v_\parallel = 0$, and the particle is reflected. A particle with small $v_\perp / v_\parallel$ at the mid plane where $B = B_0$ is a minimum can escape if the maximum field B_m is not sufficiently large. Let $B = B_0$, $v_\parallel = v_{\parallel 0}$ and $v_\perp = v_{\perp 0}$ at the mid plane. At the turning point $B = B'$, $v_\parallel = 0$, and $v_\perp = v_\perp'$. Conserve μ

$$\frac{1}{2}\frac{m v_{\perp 0}^2}{B_0} = \frac{1}{2}\frac{m v_\perp'^2}{B'}$$

Conserve energy

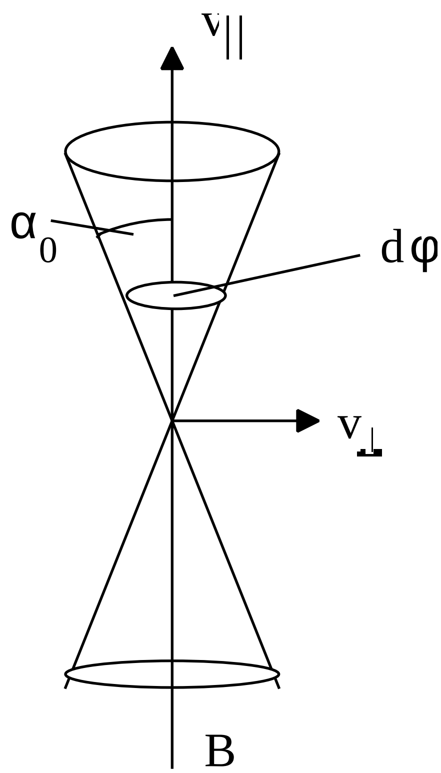
$$v_\perp'^2 = v_{\perp 0}^2 + v_{\parallel 0}^2 \equiv v_0^2$$

$$\frac{B_0}{B'} = \frac{v_{\perp 0}^2}{v_\perp'^2} = \frac{v_{\perp 0}^2}{v_0^2} = \sin^2(\theta)$$

θ is pitch angle of particle in weak field. Particles with smaller pitch θ will mirror in regions of higher B. If θ is too small $B' > B_m$ and the particle escapes. The smallest θ of a confined particle is given by the mirror ratio R_m:

$$\sin^2(\theta) = \frac{B_0}{B_m} = \frac{1}{R_m}$$

Particles which escape are said to be in the loss cone, independent of q and m. Collisions can scatter particles into loss cones. Electrons are lost easily because collision frequency is higher.



Consider the particles at the low field point where B = B_0. Let there be a uniform distribution of pitch angles α. The probability of getting lost is

$$P = \Phi/(2\pi) = \int_0^{\alpha_0} \sin(\alpha)\,d\alpha = 1 - \cos(\alpha_0)$$

$$= 1 - \left(1 - \sin^2(\alpha_0)\right)^{1/2} = 1 - \left(1 - \frac{B_0}{B_M}\right)^{1/2} = 1 - \left(1 - \frac{1}{R_M}\right)^{1/2}$$

Planetary Loss Cones exist. Take account of atmosphere. e.g. define equatorial loss cone for Earth as that pitch angle α_0 such that its mirror point is at 100 km from the surface. This is arbitrary, but used because at 100 km the density is high enough for scattering of electrons to occur. Therefore at 100 km and below a mirroring electron will probably get absorbed by the atmosphere and lost from the radiation belt. The equatorial loss cone for a dipole line of force crossing the equator at 6 R_E is about 3^0. All electrons within 3^0 equatorial pitch angle cone are precipitated because they are mirroring below 100 km. Electrons outside of the loss cone mirror at heights above 100 km are trapped radiation belt electrons.

Bounce Period in a dipole

Consider trapped particles in the Earth's dipole field. Period of north south motion of guiding center is

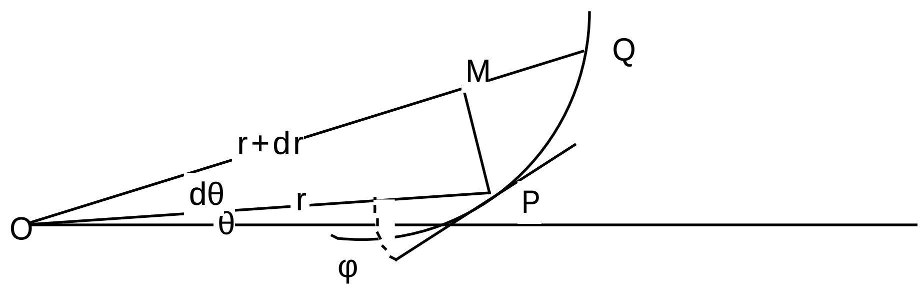
$$T_b = \int \frac{ds}{v_{\|}}$$

ds along path, $v_{\|}$ along path vector B, integral over complete period. Note

$$v_{\|}^2 = v^2 - v^2 = v^2\left(1 - \sin^2(\alpha)\right) = {}^2\left(1 - \frac{B}{B_0}\sin^2(\alpha_0)\right)$$

where conservation of the first invariant μ has been applied, $v^2 = v_0^2$ and subscript 0 means at the equator.

We need an expression for ds. Work in spherical coordinates



(r,θ) is polar coordinate of point P. PM perpendicular to OQ.

$$\sin(OQP) = r\frac{\sin(\delta\theta)}{PQ} = r\frac{\sin(\delta\theta)}{\delta\theta}\frac{\delta\theta}{\delta s}\frac{\delta s}{PQ}$$

Let Q approach P, then angle OQP becomes the angle between the tangent and the radius vector, denoted as φ. Also $\sin(\delta\theta)/\delta\theta \to 1$; $\delta s/PQ \to 1$; $\delta\theta/\delta s \to d\theta/ds$ and

$$\sin(OQP) \to \sin(\varphi) = r\frac{d\theta}{ds}$$

Similarly

$$\cos(OQP) = \frac{MQ}{PQ} = \frac{OQ - OM}{PQ}$$

$$= \frac{r + \delta r - r\cos(\delta\theta)}{\delta s}\frac{\delta s}{PQ}$$

$$= \left(\frac{r\left(1 - \cos(\delta\theta)\right)}{\delta s} + \frac{\delta r}{\delta s}\right)\frac{\delta s}{PQ}$$

i.e.

$$\cos(\varphi) = \left(0 + \frac{dr}{ds}\right)x1 = \frac{dr}{ds}$$

Then

$$\tan(\varphi) = r\frac{d\theta}{dr}$$

and since

$$\sec^2(\varphi) = 1 + \tan^2(\varphi); \quad \cos ec^2(\varphi) = 1 + \cot^2(\varphi)$$

then

$$\left(\frac{ds}{dr}\right)^2 = 1 + r^2\left(\frac{d\theta}{dr}\right)^2$$

$$\frac{1}{r^2}\left(\frac{ds}{d\theta}\right)^2 = 1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2$$

$$i.e \quad \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

For a dipole field $r = r_0\cos^2(\lambda)$, where $\lambda = \pi/2-\theta$, and finally

$$ds = r_0\cos(\lambda)\left(1 + 3\sin^2(\lambda)\right)^{1/2}d\lambda$$

Also for a dipole $\frac{B}{B_0} = \frac{\left(1 + 3\sin^2(\lambda)\right)^{1/2}}{\cos^6(\lambda)}$, so that

$$T_b = \int_0^{\lambda_{max}} \frac{r_0\cos(\lambda)\left(1 + 3\sin^2(\lambda)\right)^{1/2}}{v\left[1 - \sin^2(\alpha_0)\frac{\left(1 + 3\sin^2(\lambda)\right)^{1/2}}{\cos^6(\lambda)}\right]}d\lambda = \frac{4r_0}{v}I_1$$

$$I_1 = \int_0^{\lambda_{max}} \frac{\cos(\lambda)\left(1+3\sin^2(\lambda)\right)^{1/2}}{\left[1-\sin^2(\alpha_0)\frac{\left(1+3\sin^2(\lambda)\right)^{1/2}}{\cos^6(\lambda)}\right]} d\lambda$$

λ_1 is the mirror point of the particle in the northern hemisphere. Here $v_{||} = 0$ and λ_m is given by the solution of

$$\cos^6(\lambda_m) - \sin^2(\alpha_0)\left(1+3\sin^2(\lambda_m)\right)^{1/2} = 0$$

Now I_1 is dimensionless, and solved numerically to be

$$I_1 \approx 1.3 - 0.56\sin(\alpha_0)$$

Note $4r_0/v$ depends on the particle energy.

Now particles are precipitated (lost) if the distance of closest approach

$$\frac{R_{\min}}{r_0} < \frac{1}{L}$$

i.e. particles are trapped if the distance of closest approach is larger than R_{min}. Now the field line equation is $r = r_0\cos^2(\lambda)$, so that for particles mirroring at the plane surface we have λ_{min} given by

$$\cos^2(\lambda_m) = \frac{1}{L}$$

For Earth, $r_0 = L\,R_E$ (L is a parameter introduced by McIlwain). Figure shows bounce period as a function of electron energy for the marginally trapped electrons, for L = 1 to 8. For Auroral lines of force (L = 6), and typical electron energies of 10 keV to 50 keV, the bounce period is a few seconds.

Drift period in a dipole field

To calculate the guiding center drift period in a dipole field, remember that there are two components, the gradient drift and the curvature drift. They can be combined together as

$$v_R + v_B = \frac{m}{q}\frac{\bar{R}_c \times \bar{B}}{B^2 R_c^2}\left(\frac{v_\perp^2}{2} + v_{||}^2\right) = \frac{1}{\omega_c R_c}\frac{\bar{R}_c \times \bar{B}}{B R_c}\left(\frac{v_\perp^2}{2} + v_{||}^2\right)$$

where ω_c is the cyclotron frequency, and R_c is the radius of curvature. This drift is perpendicular to vector B and vector ρ, that is around the word (equivalent to along the equator). The expression for the radius of curvature of a line in polar coordinates is

$$\rho = \frac{\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - 2\frac{d^2 r}{d\theta^2}}$$

For the dipole field

$$r = r_0 \cos^2(\lambda); \quad \lambda = 90 - \theta$$

so that the radius of curvature vector is

$$\rho = r_0 \frac{\cos(\lambda)\left(1 + 3\sin^2(\lambda)\right)^{\frac{3}{2}}}{3\left(1 + \sin^2(\lambda)\right)}$$

The cyclotron frequency for the dipole is

$$\omega_c = \omega_0 \frac{\left(1 + 3\sin^2(\lambda)\right)^{\frac{1}{2}}}{\cos^6(\lambda)}$$

where ω_0 is the value at the equator. T hen the drift velocity at a latitude λ is written as

$$v_D = \frac{3v^2}{2} \frac{\left(1 + \sin^2(\lambda)\right)\cos^5(\lambda)}{\omega_0 r_0 \left(1 + 3\sin^2(\lambda)\right)^2} \left[2 - \sin^2(\alpha_0)\frac{\left(1 + 3\sin^2(\lambda)\right)^{\frac{1}{2}}}{\cos^6(\lambda)}\right]$$

Note we have used

$$v_\parallel = v\cos(\alpha); \quad v_\perp = v\sin(\alpha) \ (\alpha \text{ is the pitch angle})$$

$$v_\parallel^2 + \frac{1}{2}v_\perp^2 = v^2\cos^2(\alpha) + v^2\sin^2(\alpha) - \frac{1}{2}v^2\sin^2(\alpha)$$

$$= v^2\left(1 - \frac{1}{2}\sin^2(\alpha)\right) = v^2\left(1 - \frac{1}{2}\frac{B}{B_0}\sin^2(\alpha_0)\right)$$

$$= \frac{v^2}{2}\left(2 - \frac{\left(1 + 3\sin^2(\lambda)\right)^{\frac{1}{2}}}{\cos^6(\lambda)}\sin^2(\alpha_0)\right)$$

Now the angular drift (around the earth) which occurs in one bounce period is

$$\Delta\Phi = \int \frac{v_D}{r\cos(\lambda)} \frac{ds}{v_\parallel} \text{ (ds along vector B)}$$

(because bounce period is $T_b = \int ds / v_\parallel$, so distance covered in a bounce period = $T_b = \int v_D ds / v_\parallel$, and angular distance is as given. The angular drift, averaged over a bounce, is then

$$\langle\omega_D\rangle = \frac{\Delta\Phi}{2\pi T_B}$$

Using expressions derived before, we can write this as

$$\langle\omega_D\rangle = \frac{3v^2}{2\omega_0 r_0^2} \frac{I_2}{I_1} = \frac{3E}{qB_0 R_E^2 L^2} \frac{I_2}{I_1}$$

with kinetic energy E = mv^2/2, and

$$I_2 = \int_0^{\lambda_{max}} \frac{\cos^4(\lambda)\left(1+\sin^2(\lambda)\right)}{\left[1 - \sin^2(\alpha_0)\frac{\left(1+3\sin^2(\lambda)\right)^{1/2}}{\cos^6(\lambda)}\right]} \left[2 - 3\sin^2(\alpha_0)\frac{\left(1+3\sin^2(\lambda)\right)^{1/2}}{\cos^6(\lambda)}\right] d\lambda$$

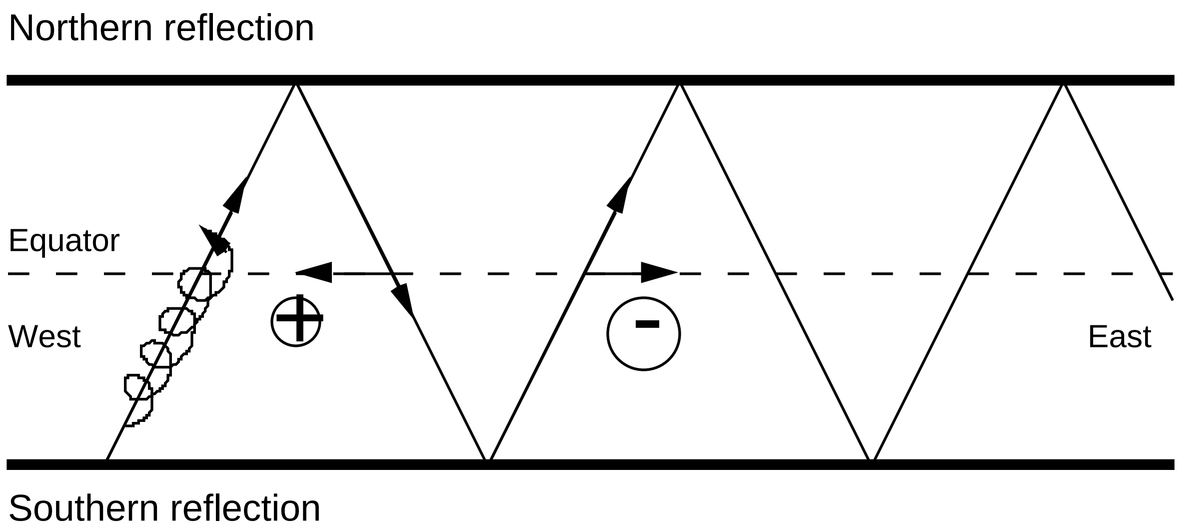
Numerically it is found that $I_2(\alpha_0)/I_1(\alpha_0) \approx 0.35 + 0.15\sin(\alpha_0)$, i.e. varying from 0.35 for a_0 = 0^0 particles to 0.5 for $a_0 = 90^0$ particles. Then the bounce averaged drift period

$$\langle T_D\rangle = \frac{1}{\langle\omega_D\rangle} \approx 50/(LE)$$

with T_D in minutes and E in MeV. Typical values (50 keV, L = 6) is about an hour.

The Ring Current around the earth

The drifting bounced orbits just discussed are represented as below in a Mercator projection. Magnetic storms are accompanied by a decrease in the horizontal field intensity at the earth. A westward current around the earth would do this. This is expected from trapped particles. Several MA are carried.



We know the drift velocity

$$\mathrm{v}_R + \mathrm{v}_{\nabla B} = \frac{m}{q}\frac{\bar{R}_c \times \bar{B}}{B^2 R_c^2}\left(\frac{\mathrm{v}^2}{2} + \mathrm{v}_\parallel^2\right) = \frac{1}{\omega_c R_c}\frac{\bar{R}_c \times \bar{B}}{B R_c}\left(\frac{\mathrm{v}^2}{2} + \mathrm{v}_\parallel^2\right)$$

from which we derived the averaged angular frequency in the equatorial direction

$$\omega_D \approx \frac{3\mathrm{v}^2}{2\omega_0 r_0^2} \approx \frac{3E}{qB_0 r_0^2}$$

i.e. the current density

$$J = nq\mathrm{v}_D \approx nqr_0\omega_D \approx -\frac{3nE}{B_0 r_0}\hat{\varphi}$$

with n the number density and E the energy of the particles. Then the total current is given by

$$I d\bar{l} = \bar{J} dV$$

i.e.

$$I \approx -\frac{3E_{tot}}{2\pi B_0 r_0^2}$$

where E_{tot} is the total energy associated with the particles. Then the change in field at the center of this loop is

$$\Delta\bar{B} = \frac{\mu_0 I}{2a} \approx -\frac{3\mu_0 E_{tot}}{4\pi B_0 r_0^3}\bar{z}$$

The total perturbation must account for another contribution, namely the diamagnetic current due to the cyclotron motion. This is calculated by noticing that the diamagnetic contribution is equivalent to a ring of dipoles of radius r and total magnetic moment μ. Because r >> r_c, the cyclotron radius, we can estimate the field at the center of the dipole ring as

(from previous notes, $B(r,\lambda) = \frac{\mu_0 M}{4\pi r^3}\left(1 + 3\sin^2(\lambda)^{1/2}\right)$)

$$\overline{B}_{diamagnetic} = -\frac{\mu_0}{4\pi r^3}\overline{\mu} = \frac{\mu_0 E_{tot}}{4\pi B_0 r_0^3}\overline{z}$$

and

$$\overline{\mu} = -\frac{mnv^2\overline{B}}{2B^2} = -E_{tot}\frac{\overline{B}}{B^2} = -\frac{E_{tot}}{B}\overline{z}$$

Note the individual dipoles are aligned with the magnetic field direction

The total perturbed field is then

$$\Delta\overline{B} \approx -\frac{3\mu_0 E_{tot}}{4\pi B_0 r_0^3}\overline{z} + \frac{\mu_0 E_{tot}}{4\pi B_0 r_0^3}\overline{z} \approx -\frac{\mu_0 E_{tot}}{2\pi B_0 r_0^3}\overline{z} = -\frac{2E_{tot}}{M}\overline{z}$$

where we have used

$$B = \frac{\mu_0 M}{4\pi r^3}$$

Non Uniform E (finite Larmor radius)

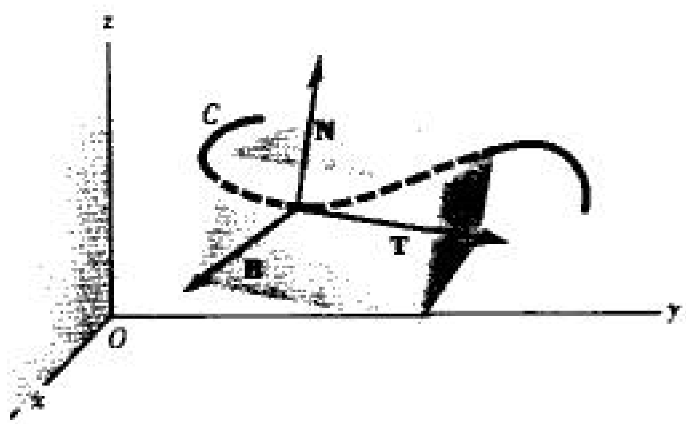
Time varying E (polarization drift)

Time Varying B (magnetic moment)

Adiabatic Invariants

Appendix - Some Geometry

If C is a space curve defined by the function $\bar{r}(u)$, then $d\bar{r}/du$ is a vector in the direction of the tangent to C. If the scalar u is the arc length s measured from some field point C, then $d\bar{r}/ds$ is a unit tangent vector to C, and is called $\bar{T}$. The rate at which $\bar{T}$ changes with respect to s is a measure of the curvature of C and is given by $d\bar{T}/ds$. The direction of dT/ds at any point on C is normal to the curve at that point. If $\bar{N}$ is a unit vector in this normal direction, it is called the unit normal. Then $d\bar{T}/ds = \kappa\bar{N}$, where κ is called the curvature, and ρ = 1/κ is called the radius of curvature.



The position vector at any point is

$$\bar{r} = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$$

Therefore

$$\bar{T} = \frac{d\bar{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$$

$$\frac{d\bar{T}}{ds} = \frac{d^2x}{ds^2}\hat{i} + \frac{d^2y}{ds^2}\hat{j} + \frac{d^2z}{ds^2}\hat{k}$$

But $\frac{d\bar{T}}{ds} = \kappa \bar{N}$; $\kappa = \left|\frac{d\bar{T}}{ds}\right|$; $\rho = \frac{1}{\kappa} = \left(\left(\frac{d^2 x}{ds^2}\right)^2 + \left(\frac{d^2 y}{ds^2}\right)^2 + \left(\frac{d^2 z}{ds^2}\right)^2\right)^{-\frac{1}{2}}$

Spherical Coordinates (r, θ, ϕ).

Transformation equations:

$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta$

where $r \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi < 2\pi$.

Scale factors: $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$

Element of arc length:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

Jacobian: $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$

Element of volume: $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Laplacian: $\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}.$